

NBB bases of some pattern avoiding lattices

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Abstract

In this paper we will determine the NBB bases with respect to standard ordering of coatoms (resp. atoms) of 123-132-213-avoiding (resp. 321-avoiding) lattices. Using these expression we will calculate the Möbius numbers of 123-132-213-avoiding lattices and 321-avoiding lattices. These values become some modification of fibonacci polynomials.

1 Introduction

In this paper we give expressions of NBB bases of 123-132-213-avoiding lattices and 321-avoiding lattices. Using these expressions we will calculate the Möbius numbers of these lattices. Surprisingly these values become some modification of fibonacci polynomials. We introduce modified fibonacci polynomials $\{F_n(q)\}_{n \in \mathbb{N}}$ later. Let A_n (resp. B_n) be the partially ordered set of the 123-132-213-avoiding (resp. 321-avoiding) permutations with the weak order on the permutation group S_n with a unique minimal element (resp. maximal element) for $n \in \mathbb{N}$. We also determine the NBB bases for A_n and B_n with respect to a natural total ordering of atoms or coatoms of them. Using the modified fibonacci polynomials and the expression of the NBB bases we will consider the Möbius numbers of A_n and B_n for each $n \in \mathbb{N}$.

Let P be a poset and $\text{Int}(P)$ the set of intervals of P . We call the function $\mu : \text{Int}(P) \rightarrow \mathbb{Z}$ the Möbius function of P if μ satisfies the following identity.

$$\sum_{x \leq y \leq z} \mu([x, y]) = \delta_{x,z} \quad (1)$$

If P has a maximum element $\hat{1}$ and a minimum element $\hat{0}$. Then we put $\mu(P) := \mu([\hat{0}, \hat{1}])$. We call $\mu(P)$ the Möbius number of P . Our main result is as following.

Theorem 1.1

For $n \in \mathbb{N}_{\geq 3}$ we have

$$\mu(A_n) = \mu(B_n) = (-1)^{F_{n-2}}(-1). \quad (2)$$

2 Preliminaries

2.1 Bounded below sets

This subsection we introduce a technique to calculate Möbius numbers of lattices which is given in Blass and Sagan's paper [2].

Throughout this subsection (L, \leq) will denote a finite lattices. We will denote it L for short. We will use \wedge for the meet (greatest lower bound) and \vee for the join (least upper bound) in L . Since L is finite it also has the unique minimal element $\widehat{0}$ and the unique maximal element $\widehat{1}$. We let $\mu(L) := \mu([\widehat{0}, \widehat{1}])$. Our goal in this subsection is to give a combinatorial description of $\mu(L)$. Let $A(L)$ (resp. $B(L)$) be the set of coatoms (resp. atoms) of L . Give $A(L)$ (resp. $B(L)$) an arbitrary total order, which we denote \trianglelefteq_A (resp. \trianglelefteq_B) to distinguish it from \leq in L . A nonempty set $D \subseteq A(L)$ (resp. $D' \subseteq B(L)$) is *bounded below* (BB for short) if for every $d \in D$ (resp. $d' \in D'$) there is an $a \in A(L)$ (resp. $a' \in B(L)$) such that $a \triangleleft_A d$ and $a > \wedge D$ (resp. $a' \triangleleft_B d$ and $a' < \vee D'$). So a (resp. a') is simultaneously a strict lower bound for d (resp. d') in the total order \trianglelefteq_A (resp. \trianglelefteq_B) and for $\wedge D$ (resp. $\vee D'$) in \leq . We will say that $B \subseteq A(L)$ (resp. $B' \subseteq B(L)$) is *NBB* if B (resp. B') does not contain any D (resp. D') which is bounded below. In this case we will call B (resp. B') an *NBB base* for $x = \wedge B$ (resp. $x' = \vee B'$). In [2] Blass and Sagan proved the following statement.

Theorem 2.1 ([2])

Let L be any finite lattice. Let $A(L)$ (resp. $B(L)$) be the set of coatoms (resp. atoms) of L and \trianglelefteq_A (resp. \trianglelefteq_B) any total order on $A(L)$ (resp. $B(L)$). Then we have

$$\mu(L) = \sum_{B \in A(L), \wedge B = \widehat{0}} (-1)^{|B|} \quad (3)$$

$$\mu(L) = \sum_{C \in B(L), \vee C = \widehat{1}} (-1)^{|C|} \quad (4)$$

where the sum is over all NBB bases of $\widehat{0}$ (resp. $\widehat{1}$) and $|\cdot|$ denotes cardinality.

2.2 Modified Fibonacci polynomials

In this subsection we introduce modified Fibonscci polynomials.

Definition 2.1

We define the sequences $\{F_n(q)\}_{n \in \mathbb{N}}$ by the following relations:

$$F_1(q) : = 1, F_2(q) := 1, \quad (5)$$

$$F_{k+2}(q) = F_{k+1}(q) + qF_k(q), \text{ for } k \geq 1. \quad (6)$$

We call the sequences $\{F_n\}_{n \in \mathbb{N}}$ modified Fibonacci polynomials.

Notation 2.1

The Fibonacci polynomials $F'_n(q)$ are defined by the following relations:

$$F'_1(q) : = 1, F'_2(q) := q, \quad (7)$$

$$F'_{k+2}(q) = qF'_{k+1}(q) + F'_k(q), \text{ for } k \geq 1. \quad (8)$$

But in this paper we don't use the Fibonacci polynomials.

Let X be a subset of $[n] := \{1, 2, \dots, n\}$. We call X a sparse set if and only if $1 \in X$ and if $i \in X$ then $i + 1 \notin X$ for $1 \leq i \leq n - 1$. For $n = 4$ the corresponding sparse sets are $\{1\}$, $\{1, 3\}$ and $\{1, 4\}$.

Then we have the following proposition. A simple calculation yields the statement of the proposition so we omit the proof.

Proposition 2.1

Let $H_n(q) := \sum_{X: \text{sparse set of } [n]} q^{\#X-1}$. Then we have

$$H_n(q) = F_n(q) \quad (9)$$

for $n \in \mathbb{N}$.

2.3 The weak order on the symmetric group

In this subsection we will introduce the weak order and its lattice structure [1] [4]. For $n \in \mathbb{N}$ let σ be an element of the permutation group S_n . We put $\text{Inv}(\sigma) := \{(i, j) \mid 1 \leq i < j \leq n, \sigma(i) > \sigma(j)\}$. We write $\sigma \leq \tau$ if $\text{Inv}(\sigma) \subset \text{Inv}(\tau)$. This determines the *weak order* on S_n . This weak order is a lattice. The identity permutation 1_n is the minimum element and $n(n-1) \cdots 21$ is the maximum element. A set J is the inversion set of a permutation in S_n if and only if both J and its complement $\text{Inv}(n(n-1) \cdots 21) - \text{Inv}(J)$ are transitively closed (i.e. $(i, j) \in J$ and $(j, k) \in J$ imply $(i, k) \in J$, and the same for its complement). The join (least upper bound) of two permutations σ and $\tau \in S_n$ is the permutation whose inversion set is the transitive closure of the union of the inversion sets of σ and τ

$$\{(i, j) \mid \exists \text{chain } i = k_0 < \cdots < k_s = j \text{ s.t. } \forall r, (k_{r-1}, k_r) \in \text{Inv}(\sigma) \cup \text{Inv}(\tau)\}. \quad (10)$$

We denote it $\sigma \vee \tau$.

Similarly, the meet (greatest lower bound) of σ and τ is the permutation whose inversion set is

$$\{(i, j) \mid \forall \text{chains } i = k_0 < \cdots < k_s = j, \exists r \text{ s.t. } (k_{r-1}, k_r) \in \text{Inv}(\sigma) \cap \text{Inv}(\tau)\}. \quad (11)$$

3 The case of 123-132-213 avoiding lattices

For each $n \in \mathbb{N}$ we define A'_n to be the partially ordered set of 123-132-213-avoiding permutations associated with the weak order on S_n . We put $A_n := A'_n \cup \{\widehat{0}\}$ where $\widehat{0}$ is a unique minimum element. For example we have $A_1 = \{1, \widehat{0}\}$, $A_2 = \{12, 21, \widehat{0}\}$ and $A_3 = \{231, 312, 321, \widehat{0}\}$. Let X be a subset of permutations of S_n . Put $(n+1)X := \{(n+1)a_1 \cdots a_n \mid a_1 \cdots a_n \in X\}$. The following theorem is known.

Theorem 3.1 ([3])

Let \tilde{A}_n be the set of 123-132-213-avoiding permutations in S_n for $n \in \mathbb{N}$. (We will consider \tilde{A}_n as a set.) Then we have

$$\tilde{A}_{n+2} = (n+1)(n+2)\tilde{A}_n \uplus (n+2)\tilde{A}_{n+1} \quad (12)$$

From Theorem 3.1 we have the following lemma.

Lemma 3.1

The poset A_n is an order filter of S_n for $n \in \mathbb{N}$.

PROOF

We will prove by induction on n . We assume that this lemma holds for $\leq n$. Let $\sigma \in A_{n+1}$. Then we have $\sigma = n(n+1)a_1 \cdots a_{n-1}$ with $a_1 \cdots a_{n-1} \in A_{n-1}$ or $\sigma = (n+1)b_1 \cdots b_n$ with $b_1 \cdots b_n \in A_n$ by Theorem 3.1.

The case of $\sigma = n(n+1)a_1 \cdots a_{n-1}$ with $a_1 \cdots a_{n-1} \in A_{n-1}$. For τ with $\tau \geq \sigma$ in the weak order on S_{n+1} , we have either $\tau = n(n+1)a'_1 \cdots a'_{n-1}$ or $\tau = (n+1)na''_1 \cdots a''_{n-1}$ with $a'_1 \cdots a'_{n-1}, a''_1 \cdots a''_{n-1} \geq a_1 \cdots a_{n-1}$. By assumption we have $a'_1 \cdots a'_{n-1}, a''_1 \cdots a''_{n-1} \in A_{n-1}$. Hence we have $\tau \in A_{n+1}$.

The case of $\sigma = (n+1)b_1 \cdots b_n$ with $b_1 \cdots b_n \in A_n$. For τ with $\tau \geq \sigma$ in the weak order on S_{n+1} we have $\tau = (n+1)b'_1 \cdots b'_n$ with $b'_1 \cdots b'_n \geq b_1 \cdots b_n$. By assumption we have $b'_1 \cdots b'_n \in A_n$. Hence we have $\tau \in A_{n+1}$.

This completes the proof of our lemma. \square

Next we define

$$C_n := \{c_1 c_2 \cdots c_k \mid c_i = 1 \text{ or } 2 \text{ with } c_1 + c_2 + \cdots + c_k = n\} \cup \{\widehat{0}\} \quad (13)$$

with covering relations as following;

$$c_1 \cdots c_{i-1} 2 c_{i+1} \cdots c_k \prec c_1 \cdots c_{i-1} 1 c_{i+1} \cdots c_k, \quad (14)$$

where $\widehat{0}$ the minimum elements for $n \in \mathbb{N}$. Then C_n has a poset structure for $n \in \mathbb{N}$.

Proposition 3.1

For $n \in \mathbb{N}$ we have $A_n \simeq C_n$ as a poset.

Proof

For each $n \in \mathbb{N}$ we will define the map $\phi_n : C_n \rightarrow A_n$ by induction.

We put $\phi_1(1) := 1$, $\phi_1(\widehat{0}) := \widehat{0}$, $\phi_2(11) := 21$, $\phi_2(2) := 12$ and $\phi_2(\widehat{0}) := \widehat{0}$.

For $n \in \mathbb{N}$, $c_1 \cdots c_k \in C_n$ we define ϕ_n as follows;

$$\phi_n(c_1 \cdots c_k) := n\phi_{n-1}(c_2 \cdots c_k) \quad \text{if } c_1 = 1 \quad (15)$$

$$\phi_n(c_1 \cdots c_k) := (n-1)n\phi_{n-2}(c_2 \cdots c_k) \quad \text{if } c_1 = 2 \quad (16)$$

$$\phi_n(\widehat{0}) := \widehat{0}. \quad (17)$$

For example we have $\phi_3(\widehat{0}) = \widehat{0}$, $\phi_3(111) = 321$, $\phi_3(21) = 231$ and $\phi_3(12) = 312$.

Next we will define the map $\psi_n : A_n \rightarrow C_n$ for each $n \in \mathbb{N}$ by induction.

We put $\psi_1(1) := 1$, $\psi_1(\widehat{0}) := \widehat{0}$, $\psi_2(21) := 11$, $\psi_2(12) := 2$ and $\psi_2(\widehat{0}) := \widehat{0}$.

For $n \in \mathbb{N}$ we define ψ_n as follows;

$$\psi_n((n-1)na_1 \cdots a_{n-2}) := 2\psi_{n-2}(a_1 \cdots a_{n-2}) \quad (18)$$

$$\psi_n(na_1 \cdots a_{n-1}) := 1\psi_{n-1}(a_1 \cdots a_{n-1}) \quad (19)$$

$$\psi_n(\widehat{0}) := \widehat{0}. \quad (20)$$

For example we have $\psi_3(\widehat{0}) = \widehat{0}$, $\psi_3(321) = 111$, $\psi_3(231) = 21$ and $\psi_3(312) = 12$. Note that if $a_1 \cdots a_n \in A_n$ we have either $a_1 = n-1$ and $a_2 = n$ or $a_1 = n$. It is easy to see that $\psi_n \circ \phi_n = id_{C_n}$ and $\phi_n \circ \psi_n = id_{A_n}$.

Next we have to show that the map ϕ_n and ψ_n are both order preserving.

The case of ϕ_n . We will show that ϕ_n preserves the covering relation of C_n . For $c_1 \cdots c_{i-1}2c_{i+1} \cdots c_k$, $c_1 \cdots c_{i-1}11c_{i+1} \cdots c_k \in C_n$ we have $c_1 \cdots c_{i-1}2c_{i+1} \cdots c_k \prec c_1 \cdots c_{i-1}11c_{i+1} \cdots c_k$. We have $\phi_n(c_1 \cdots c_{i-1}11c_{i+1} \cdots c_k) = \phi_{n-2-x}^x(c_1 \cdots c_{i-1})(x+2)(x+1)\phi_n(c_{i+1} \cdots c_k)$ and $\phi_n(c_1 \cdots c_{i-1}2c_{i+1} \cdots c_k) = \phi_{n-2-x}^x(c_1 \cdots c_{i-1})(x+1)(x+2)\phi_n(c_{i+1} \cdots c_k)$ where $x = c_{i+1} + \cdots + c_k$ and $\phi_{n-2-x}^x(c_1 \cdots c_{i-1}) = (c'_1 + x) \cdots (c'_{i-1} + x)$ for $\phi_{n-2-x}(c_1 \cdots c_{i-1}) = c'_1 \cdots c'_{i-1}$. It is easy to see that $\phi_{n-2-x}^x(c_1 \cdots c_{i-1})(x+2)(x+1)\phi_n(c_{i+1} \cdots c_k) \prec \phi_{n-2-x}^x(c_1 \cdots c_{i-1})(x+1)(x+2)\phi_n(c_{i+1} \cdots c_k)$ in A_n .

The case of ψ_n . We will show that ψ_n preserves the covering relation of A_n . Let $a_1 \cdots a_n \in A_n$ with $a_i = m$ and $a_j = m+1$ for $i < j$. Because $a_1 \cdots a_n$ avoids 213 pattern and 132 pattern, we have $j = i+1$. We have $a_1 \cdots a_{i-1}m(m+1)a_{i+2} \cdots a_n \prec a_1 \cdots a_{i-1}(m+1)ma_{i+2} \cdots a_n$. Then we have $\phi_n(a_1 \cdots a_{i-1}m(m+1)a_{i+2} \cdots a_n) = \phi_{i-1}(st(a_1 \cdots a_{i-1}))2\phi_{n-1-x}(st(a_{i+2} \cdots a_n))$ and $\phi_n(a_1 \cdots a_{i-1}(m+1)ma_{i+2} \cdots a_n) = \phi_{i-1}(st(a_1 \cdots a_{i-1}))11\phi_{n-1-x}(st(a_{i+2} \cdots a_n))$ where $st(a_1 \cdots a_{i-1}) \in S_{i-1}$ is the unique permutation $\sigma \in S_{i-1}$ such that $\sigma_s < \sigma_t \Leftrightarrow a_s < a_t$. It is easy to see that $\phi_{i-1}(st(a_1 \cdots a_{i-1}))2\phi_{n-1-x}(st(a_{i+2} \cdots a_n)) \prec \phi_{i-1}(st(a_1 \cdots a_{i-1}))11\phi_{n-1-x}(st(a_{i+2} \cdots a_n))$.

This completes the proof of our proposition. \square

Next we calculate the Möbius numbers of C_n for each $n \in \mathbb{N}$.

Let $A(C_n) := \{21 \cdots 1, 121 \cdots 1, \dots, 1 \cdots 21, 1 \cdots 12\}$ be the set of coatoms of C_n where $\sharp A(C_n) = n-1$. We give $A(C_n)$ a total order \triangleleft as follows,

$$21 \cdots 1 \triangleleft 121 \cdots 1 \triangleleft \dots \triangleleft 1 \cdots 21 \triangleleft 1 \cdots 12. \quad (21)$$

We put $\theta_i := 1 \cdots 1 \underbrace{2}_{i\text{-th}} 1 \cdots 1$. Then the following lemma is easy to prove so we will omit the proof.

Lemma 3.2

For $1 \leq i \leq n - 2$ we have

$$\theta_i \wedge \theta_{i+1} = \widehat{0}. \quad (22)$$

Lemma 3.3

We let $\{i_1, \dots, i_k\} \subset [n - 1]$ with $i_1 < \dots < i_k$. If $i_{p+1} > i_p + 1$ for $1 \leq p \leq k - 1$ then we have

$$\theta_{i_1} \wedge \theta_{i_2} \wedge \dots \wedge \theta_{i_k} = 1 \cdots \underbrace{2}_{j_1} \cdots \underbrace{2}_{j_2} \cdots \underbrace{2}_{j_k} \cdots 1, \quad (23)$$

where $j_1 = i_1$, $j_2 = i_2 - 1$, $j_3 = i_3 - 2$, \dots , $j_k = i_k - k + 1$.

Proof

For $j \geq i + 2$ we have

$$a_1 \cdots a_{i-1} \underbrace{2}_{i\text{-th}} 1 \cdots 1 \wedge a_1 \cdots a_{i-1} 1 \cdots \underbrace{2}_{j\text{-th}} \cdots 1 = a_1 \cdots a_{i-1} \cdots \underbrace{2}_{i\text{-th}} \cdots \underbrace{2}_{(j-1)\text{th}} \cdots 1$$

for $a_1, \dots, a_{i-1} \in \{1, 2\}$. From this fact and using induction we obtain the desired result. \square

Lemma 3.4

Put $X := \{\theta_{i_1}, \dots, \theta_{i_k}\} \subset A(C_n)$ with $i_{p+1} > i_p + 1$ for $1 \leq p \leq k - 1$. Then the set X is not BB with respect to our total ordering \triangleleft . Moreover X is NBB.

Proof

We have $\wedge X = 1 \cdots \underbrace{2}_{j_1} 1 \cdots \underbrace{2}_{j_2} \cdots 1 \underbrace{2}_{j_k} \cdots 1$ where $j_1 = i_1, j_2 = i_2 - 1, \dots, j_k = i_k - k + 1$. Then we have $\{y \in A(C_n) \mid y \leq \wedge X\} = \{\theta_{i_1}, \dots, \theta_{i_k}\}$. This yields that X is not BB. For $Y \subset X$ the same argument yields that Y is not BB. Hence we obtain the derived result. \square

Theorem 3.2

Let $X := \{\theta_{i_1}, \dots, \theta_{i_k}\}$ be a subset of $A(C_n)$ where $i_1 < \dots < i_k$. Then X is an NBB base of $\widehat{0}$

$\iff X$ satisfies

1. $i_1 = 1$,
2. $\{i_2 - 1, i_3 - 1, \dots, i_k - 1\}$ is a sparse set in $[n - 2]$.

Proof

(\implies)

If $X := \{\theta_{i_1}, \dots, \theta_{i_k}\}$ is an NBB base of $\widehat{0}$. For each $1 \leq i \leq n - 1$ we have $a_i > \wedge X = \widehat{0}$. So we have $\theta_1 \in X$ because X is BB. On the other hand because $\theta_{i_1} \wedge \dots \wedge \theta_{i_k} = \widehat{0}$ there exists $1 \leq j \leq k - 1$ such that $i_{j+1} = i_j + 1$. Then we have $\theta_{i_j} \wedge \theta_{i_{j+1}} = \widehat{0}$ and $\{\theta_{i_j}, \theta_{i_{j+1}}\} \subset X$ is not BB. Hence we have $j = 1$, $i_1 = 1$ and $i_2 = 2$.

If there exists $2 \leq j' \leq k-1$ such that $i_{j'+1} = i_{j'} + 1$. We put $Z := \{\theta_{i_{j'}}, \theta_{i_{j'+1}}\}$. Then $\theta_{i_{j'}} \wedge \theta_{i_{j'+1}} = \widehat{0}$ and Z is not BB. This yields that $\theta_1 \in Z$. This contraradicts the assumption $j' \geq 2$. This completes the proof of " \implies " part.

(\Leftarrow)

Let X be a subset of $[n-1]$ satisfying the above conditions. For any subset Y of X we will show that Y is not BB. We put $Y := \{\theta_{j_1}, \dots, \theta_{j_l}\} \subset X$.

When $j_1 = 1$ and $j_2 = 2$ we have $\wedge Y = \widehat{0}$ and $\theta_1 \in Y$. Hence Y is not BB. When $j_1 = 1$ and $j_2 \geq 3$ it is easy to see that Y is not BB. When $j_1 \geq 2$ it is also trivial from Lemma 3.4. This completes the proof of our statement. \square

From Theorem 2.1 and Theorem 3.2 we obtain the following result.

Theorem 3.3

We have

$$\mu(A_n) = \mu(C_n) = \sum_{X \subset [n-2] \text{ sparse set}} (-1)^{|X|+1} = (-1)F_{n-2}(-1). \quad (24)$$

4 The case of 321-avoiding lattices

For each $n \in \mathbb{N}$ we define B'_n to be the partially ordered set of 321 avoiding permutations associated with the weak order on S_n . We put $B_n := B'_n \cup \{\widehat{1}\}$ where $\widehat{1}$ is a unique maximum element. For example we have $B_1 = \{1, \widehat{1}\}$, $B_2 = \{12, 21, \widehat{1}\}$ and $B_3 = \{123, 213, 132, 312, 231, \widehat{1}\}$. Lemma 4.1 and Lemma 4.2 are trivial from the definition of B_n .

Lemma 4.1

For each $n \in \mathbb{N}$ our poset B'_n is an order ideal of S_n . Therefore B_n is a lattice.

Lemma 4.2

For each $n \in \mathbb{N}$ we have

$$\sigma \in B'_n \iff \text{If } (i, j) \in \text{Inv}(\sigma) \text{ then for any } k \text{ we have } (j, k) \notin \text{Inv}(\sigma). \quad (25)$$

We put $\sigma_i := (i, i+1)$. Let $A(B_n)$ be the set of atoms of B_n . Note that $A(B_n) = \{\sigma_1, \sigma_2, \dots, \sigma_{n-1}\}$. We define $A(B_n)$ a total order \triangleleft as following;

$$\sigma_1 \triangleleft \sigma_2 \triangleleft \dots \triangleleft \sigma_{n-1}. \quad (26)$$

Lemma 4.3

We have $\sigma_i \vee \sigma_{i+1} = \widehat{1}$ for $1 \leq i \leq n-2$.

Proof

We fix $1 \leq i \leq n-2$. We put $\tau := \sigma_i \vee \sigma_{i+1}$ in S_n with the weak order. Then we have $\tau(i) > \tau(i+1) > \tau(i+2)$. So we have $\tau \notin B'_n$. From Lemma 4.1 we obtain the derived result. \square

Lemma 4.4

We assume $1 \leq j_1 < \dots < j_l \leq n-1$ and $2 \leq j_{p+1} - j_p$ for $1 \leq p \leq l-1$. We put $Y := \{\sigma_{j_1}, \dots, \sigma_{j_l}\} \subset A(B_n)$. Then we have $\vee Y = \sigma_{j_1} \sigma_{j_2} \dots \sigma_{j_l}$.

Proof

For $1 \leq p \leq q \leq l$ we have $\sigma_{j_p} \sigma_{j_q} = \sigma_{j_q} \sigma_{j_p}$. Hence we have $\vee Y = \sigma_{j_1} \sigma_{j_2} \dots \sigma_{j_l}$ in the weak order. It is easy to see that $\sigma_{j_1} \sigma_{j_2} \dots \sigma_{j_l}$ is a 321 avoiding permutation. Hence we have $\sigma_{j_1} \sigma_{j_2} \dots \sigma_{j_l} \in B'_n$. \square

Next we will determine the NBB bases with respect to \triangleleft .

Theorem 4.1

Put $X := \{\sigma_{i_1}, \dots, \sigma_{i_k}\} \subset A(B_n)$ where $1 \leq i_1 < \dots < i_k \leq n-1$. We assume that $\vee X = \hat{1}$.

Then we have X is NBB \iff

1. $i_1 = 1$
2. $\{i_2 - 1, i_3 - 1, \dots, i_k - 1\}$ is a sparse set of $[n-2]$.

Proof

(\implies)

Because X is an NBB base of $\hat{1}$, we have $\sigma_1 \in X$. Hence we have $i_1 = 1$. We put $Y := \{\sigma_{i_2}, \sigma_{i_3}, \dots, \sigma_{i_k}\}$. If $\vee Y = \hat{1}$ then we have $\sigma_1 < \hat{1}$ and $\sigma_1 \notin Y$. Then we have that Y is not BB. This contradicts the assumption that X is an NBB base. So we have $\vee Y \neq \hat{1}$. By Lemma 4.3 we have $i_2 + 1 < i_3, i_3 + 1 < i_4, \dots, i_{k-1} + 1 < i_k$. If $i_2 \neq 2$ we have $\vee X \neq \hat{1}$. Hence we have that the set $\{i_2 - 1, i_3 - 1, \dots, i_k - 1\}$ is a sparse set of $[n-2]$.

(\impliedby)

Because σ_1 and σ_2 are elements of X , we have $\vee X = \hat{1}$. Let Y be a subset of X . We put $Y := \{\sigma_{m_1}, \dots, \sigma_{m_p}\} \subset X$ with $1 \leq m_1 < m_2 < \dots < m_p \leq n-1$. We have to show that Y is not a BB base. If $m_1 = 1$ it is clear. If $m_1 \neq 1$ we have $m_1 + 1 < m_2, m_2 + 1 < m_3, \dots, m_{p-1} + 1 < m_p$. By Lemma 4.4 we have $\vee Y = \sigma_{m_1} \sigma_{m_2} \dots \sigma_{m_p}$. Hence we have $\{x \mid x \in A(B_n), x \leq \vee Y\} = \{m_1, m_2, \dots, m_p\}$. Hence we have that Y is not a BB base. This completes the proof of our result. \square

From Theorem 2.1 and Theorem 4.1 we obtain the following result.

Theorem 4.2

We have

$$\mu(B_n) = \sum_{X \subset [n-2] \text{ sparse set}} (-1)^{|X|+1} = (-1)F_{n-2}(-1). \quad (27)$$

Notation 4.1

Note that For each $n \in \mathbb{N}$ the Tamari lattice T_n is the poset of 132 avoiding permutations with weak order on S_n . It is well known that for each $\sigma \in S_3 \setminus \{123, 321\}$ the poset of σ avoiding permutations is also the Tamari lattice.

Acknowledgement

The author wishes to thank Professor Jun Morita for his valuable advices.

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